Definition. Orthogonality $\langle u, v\rangle = 0 \Leftrightarrow u \perp v$.

Definition. $\{\varphi_i\}_{i\in I}$, I countable is **orthonormal** if $\|\varphi_i\|_H = 1$, and $\langle \varphi_i, \varphi_j\rangle = 0, i = j$.

Example. Basis vectors in $\mathbb{C}^n$.

Definition. Fourier Coefficients if $f \in H$ the (discrete) Fourier Coefficients of f are

$$c_n(f) = \langle f, \varphi_n\rangle \in \mathbb{C}, \qquad n \in I$$

Assume $I = \{1, \dots\}$ is enumerated. Consider

$$S_N(f) = \sum_{n=1}^{N} c_n(f)\varphi_n(f)$$

Theorem. ***Bessel's Inequality*** *If $\{\varphi_i\}_{i\in I}$ is an orthonormal set and $f \in H$ a Hilbert space, then*

$$\sum_{n=1}^{\infty} |c_n(f)|^2 \leq \|f\|^2 < \infty$$

Proof. $S_N = \sum_{i=1}^{N} c_i(f)\varphi_i,\ c_i(f) = \langle f, \varphi_i\rangle$. Recall that

$$f, g \in H,\ f \perp g \Rightarrow \|f+g\|^2 = \langle f+g, f+g\rangle = \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle = \|f\|^2 + \|g\|^2$$

Apply this repeatedly so

$$\|S_N(f)\|^2 = \left\|\sum_{i=1}^{N-1} c_i(f)\varphi_i + c_N(f)\varphi_N\right\|^2 = \sum_{i=1}^{N} \|c_i(f)\varphi_i\|^2 = \sum_{i=1}^{N} \|c_i(f)\|^2$$

Consider $f - S_N(f) \perp \varphi_i,\ 1 \le i \le N$. Then

$$\langle f - S_N(f), \varphi \rangle = \langle f, \varphi_j \rangle - \langle S_N(f), \varphi_j \rangle = c_j(f) - \left\langle \sum_{n=1}^{N} c_n \varphi_n, \varphi_j \right\rangle = 0$$

So $\langle f - S_N(f), S_N(f) \rangle = 0$ then

$$\begin{aligned}\|f\|^2 = \|f - S_N(f) + S_N(f)\|^2 &= \|f - S_N(f)\|^2 + \|S_N(f)\|^2 \Rightarrow \\ = \|S_N(f)\|^2 = \sum_{i=1}^{N} \|c_i(f)\|^2 &\le \|f\|^2, \qquad \text{because } \|f - S_n\|^2 \ge 0\end{aligned}$$

□

If $M \ge N$ then

$$\|S_N(f) - S_M(f)\|^2 = \left\| \sum_{n=N}^{M} c_n(f) \varphi_n \right\|^2 = \sum_{n=N}^{M} |c_n(f)|^2$$

so it follows that $S_N(f)$ is Cauchy in H. Then the completeness of H implies that $S_N(f) \to S(f)$ converges in H. Does $S(f) = f$ always? We have a condition below to answer this.

Definition. In H $\{\varphi_i\}_{i \in I}$, I countable is said to be **complete** if, $w \in H$ and $\langle w, \varphi_i \rangle = 0,\ \forall i \in I$ then $w = 0$.

Proposition. *Completeness of φ_i is equivalent to the condition*

$$f = \sum_{n \in I} c_n(f) \varphi_n,\ \forall f \in H, c_n = \langle f, \varphi_n \rangle$$

i.e. $f = \lim_{N \to \infty} S_N(f) = S(f)$.

Proof. Look at $f - S(f)$. Then $f - S_N(f) \perp \varphi_n,\ \forall N \ge n$. We know by Cauchy-ness that $S_N(f) \to S(f)$ for some $S(f)$. Now we have to show that

$$0 = \langle f - S_N(f), \varphi_n \rangle \to \langle f - S(f), \varphi_n \rangle$$

we must simply show that $\langle S_N(f), \varphi_n \rangle \to \langle S(f), \varphi_n \rangle$. But if $g_j \to g$ in H then $\forall w \in H$, $\langle g_j, w \rangle \to \langle g, w \rangle$ since

$$|\langle g - g_j, w \rangle| \le \|g - g_j\| \|w\| \to 0.$$

So by $\langle f - S(f), \varphi_n \rangle = 0$ for all n, so by completeness, $f - S(f) = 0$ and $f = S(f)$. □

Our main example of complete orthonormal function are the fourier series. Fourier series, functions on $[-\pi, \pi] \in \mathbb{R}$. We claim that

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} \exp(inx), \qquad n \in \mathbb{Z}$$

is a complete orthonormal set in $L^2([-\pi,\pi],\mu_{Leb})$. We know that

$$\|\varphi_n\|^2 = \int |\varphi_n|^2 d\mu_{Leb} = \int_{[-\pi,\pi]} \frac{1}{2\pi} dx = 1$$

Also,

$$\langle \varphi_n, \varphi_m \rangle = \int_{[-\pi,\pi]} \varphi_n \overline{\varphi_m} d\mu_L = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{i(n-m)x} dx$$

assume $n \neq m$, then the above is

$$\frac{1}{2\pi} \int_{[-\pi,\pi]} \frac{\frac{d}{dx} e^{i(n-m)x}}{i(n-m)} dx = \frac{1}{2\pi} \frac{e^{i(n-m)x}}{i(n-m)} \Big|_{-\pi}^{\pi} = 0$$

but there is a problem: these are Riemann integrals. We have only proved the FTC applies to Riemann integrals. We would like

Theorem. *If f is continuous on $[a,b]$ then*

$$\textit{Riemann Integral } = \int_a^b f dx = \int_{[a,b]} f d\mu_{Leb} \ = \textit{Lebesgue Integral}$$

This theorem would imply that $\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_n$ is orthonormal in $L^2([-\pi,\pi])$. However, it is hard to prove that φ_n are complete.

Theorem. *Separable Hilbert space has a complete orthonormal set (called a basis)*

Proof. H is separable if it has a countable dense subset $E \subset H, \overline{E} = H$. We prove it has a basis by using Gram-Schmidt process. Order the set $E = \{e_1, e_2, \dots\}$. We have the following two steps

1. e_1 if $e_1 = 0$, then go to step 2. If $e_1 \neq 0$ then

$$\varphi_1 = \frac{e_1}{\|e_1\|}$$

2. Assume after n steps that we have $\{\varphi_1, \dots, \varphi_n\}$ orthonormal and $e_j \in \text{span}_{\mathbb{C}}\{\varphi_1, \dots, \varphi_n\}$, $j \leq n$. Then consider e_{n+1}. If $e_{n+1} \in \text{span}_{\mathbb{C}}\{\varphi_1, \dots, \varphi_n\}$ pass to next step.

   If $e_{n+1} \notin \text{span}_{\mathbb{C}}\{\varphi_1, \dots, \varphi_n\}$ then

$$g = e_{n+1} - \sum_{i=1}^{N} \langle e_{n+1}, \varphi_i \rangle \varphi_i \neq 0$$

   (because if e_{n+1} can be written as such a sum then e_{n+1} is in the span). Now $g \perp \varphi_i, 1 \leq i \leq n$, and define

$$\varphi_{n+1} = \frac{g}{\|g\|}$$

if we continue this indefinitely, then we get $\{\varphi_i\}_{\{i\in I\}}$ orthonormal.

Show completeness: Suppose $w \in H$, $w \perp \varphi_i, \forall i$. The density of E in H implies that given $\epsilon > 0$, $\exists e_l \in E$ such that $\|w - e_l\| < \epsilon$. But e_l is a finite linear combination $e_1 = \sum_{finite} c_i\varphi_i$. So $w \perp e_l$, and

$$\epsilon^2 \geq \|w - e_l\|^2 = \|w\|^2 + \|e_l\|^2 \Rightarrow \|w\| < \epsilon$$

since this holds $\forall\epsilon$, $\|w\| = 0$. □

Now lets go back and prove that the Riemann integral is the same as the Lebesgue integral (in the cases we care about)

Theorem. *If $f \in C([a,b])$ then $f \in \mathcal{L}^1([a,b])$ and*

$$\int_a^b f(x)dx = \int_{[a,b]} f d\mu_L$$

is true if f is Riemann integrable

Proof. f Riemann integrable means that f is bounded and

$$\sup_P L(f,P) = \underline{\int_a^b} f dx = \overline{\int_a^b} f dx = \inf_P U(f,P)$$

where $L(f,P)$ and $U(f,P)$ are the familiar upper and lower sums. Then set

$$s_L(P) = \sum \inf_{(a-1,a_i]} f\chi_{[a_i,a_i)}$$

a simple measurable function. Just to make sure everything is kosher, replace f by $f + c \geq 0$, so that we know we are working with positive functions. Then

$$L(f,P) = I(s_L(P)) \leq \int_{[a,b]} f d\mu_L \Rightarrow \underline{\int_a^b} f dx \leq \int_{[a,b]} f d\mu_L$$

We can prove the inequality the other way by considering $(-f + c)$ □

So now we have justified the computation of $\langle\varphi_n, \varphi_m\rangle$ where $\varphi_n = \frac{1}{\sqrt{2\pi}}e^{-inx}$